Matrix Algebra & Advanced Calculus II

Tutorial Slides

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Tutorial Content

Multivariable Functions

Concept: Multivariable Function

A multivariable scalar function is any function that maps a vector input to a scalar output. Let $f(\vec{x})$ be a multivariable function with n variables. Then:

$$f : \mathbb{R}^n \to \mathbb{R}$$

This means that f takes in a vector $\vec{x}$, and returns a scalar. Multivariable functions encode a lot of information and are seen practically everywhere.

Level Sets

Definition: The Level Set of $f(\vec{x})$

A level set of a multivariable function $f(\vec{x})$ is defined as the set of all $\vec{x}$ such that $f(\vec{x}) = c$, where $c \in \mathbb{R}$.

$$S = \{\vec{x} : f(\vec{x}) = c, c \in \mathbb{R}, \forall \vec{x} \in \mathbb{R}^n\}$$

It can be thought of as the cross section of a function and a codim 1 horizontal hyperplane to $\mathbb{R}^n$.

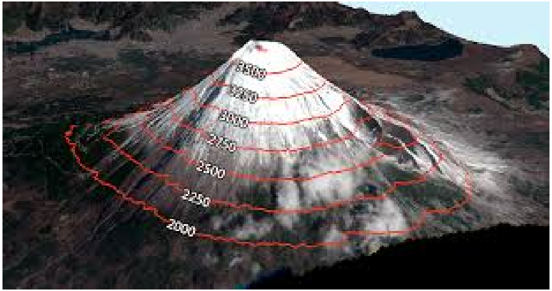


Figure: Level sets on a mountain (GISGeography)

Level Sets: Practice Problems

1) For the following functions, parameterize the level set for $f(x, y) = c, c \in \mathbb{R}$ with a single parameter t.

- $f(x, y) = x^2 + y^2$
- $f(x, y) = x^2 - y^2$
- $f(x, y) = x^{\sin(xy)}$

2) What does the level set of $f(x, y) = \sin(x) + \sin(y) = 0$ look like? How would you express this parametrically? Recall: $\sin(x) = \sin(x + 2k\pi) = \sin(\pi - x + 2k\pi)$ where $k \in \mathbb{Z}$.

3) Find any $f(x, y)$ such that the level set $f(x, y) = 0$ is the set of all pairs of odd integers. Can you do this for all pairs of squares (x, y) where $x, y = c^2, c \in \mathbb{Z}$?

Multivariable Continuity, and the Multivariable Limit

Concept: Multivariable Limit

We know that for a single-variable real valued function $f : \mathbb{R} \to \mathbb{R}$, the limit as f approaches a point a is written as:

$$\lim_{x \to a} f(x) = L$$

For a multivariable real function $g : \mathbb{R}^n \to \mathbb{R}$, the limit as g approaches a point $\vec{b}$ is written as:

$$\underbrace{\lim_{\vec{x} \to \vec{b}} f(\vec{x}) = L}_{\text{Multivariable Case}} \quad \text{or} \quad \underbrace{\lim_{(x,y) \to (a,b)} f(x, y) = L}_{\text{2-variable case}}$$

Computing Multivariable Limits

While with single variable functions one can approach a point a from the positive or negative direction (two possibilities), one can approach a multivariable point $\vec{b}$ from a potentially infinite number of directions on an infinite number of trajectories!

This means that for some specific cases (discontinuous functions) there potentially exists multiple values for limits depending on which direction you come from. This isn't a problem though!

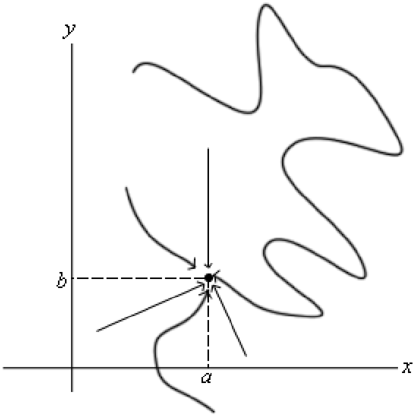


Figure: (Paul's Online Math Notes)

Continuity of Multivariable Functions

Definition: Multivariable Continuity

A multivariable function $f(\vec{x})$ is **continuous** at a point $(\vec{a})$ if:

$$\lim_{\vec{a}\to\vec{x}} f(\vec{x}) = f(\vec{a})$$

In most cases, this means that substituting the limit vector $\vec{a}$ directly into $\vec{x}$ is sufficient to computing the limit. However, some discontinuous functions do not have limits at these discontinuities.

Multivariable Discontinuity

The limit of a multivariable function $f(\vec{x})$ does not exist at a point $\vec{a}$ if two different approach paths yield different values for the limit.

Epsilon-Delta Definition of Multivariable Continuity

Definition: (ϵ-δ) Multivariable Continuity

A multivariable function $f : \mathbb{R}^n \to \mathbb{R}$ is **continuous** at a point $(\vec{a})$ if:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall \vec{x} \in \mathbb{R}^n, d(\vec{a}, \vec{x}) < \delta \implies |f(\vec{a}) - f(\vec{x})| < \varepsilon$$

In Simpler Terms: The epsilon-delta limit exists at the point $\vec{a}$ and is equal to $f(\vec{a})$.

Limits & Continuity Practice Problems

Verify if the following limits exist. If not, demonstrate that the multivariable function is discontinuous at the limit point:

- $\lim_{(x,y,z)\to(2,1,-1)} 3x^2z + yx\cos(\pi x - \pi z)$
- $\lim_{(x,y)\to(1,1)} \dfrac{2x^2 - xy - y^2}{x^2 - y^2}$
- $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{\sin(x^2 + y^2)}$
- $\lim_{(x,y)\to(0,0)} \dfrac{x^3y}{x^6 + y^2}$

Partial Derivatives

Definition: (Partial Derivative of a Multivariable Function at a Point)

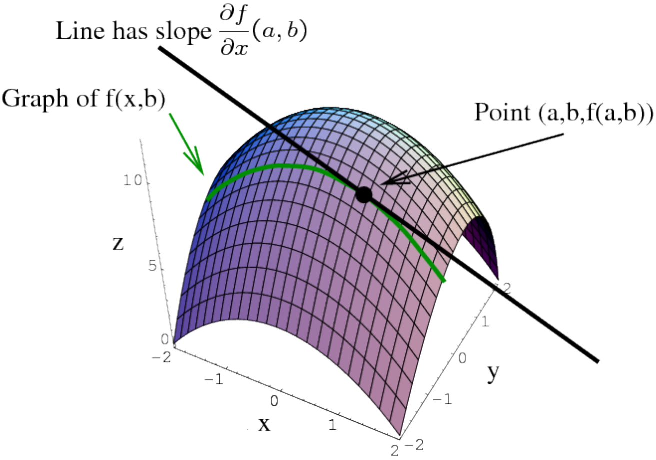
The partial derivative of an n-variable function $f(x_1, x_2, \ldots, x_n) = f(\vec{x})$ at a point $\vec{a} \in \mathbb{R}^n$ with respect to the variable x_i, where $i = 1, 2, \ldots, n$ can be defined formally by the limit:

$$\frac{\partial}{\partial x_i} f(\vec{a}) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h} = \frac{f(\vec{a} + h\hat{e}_i) - f(\vec{a})}{h}$$

Where $\hat{e}_i$ is the i-th unit vector, $\hat{e}_i \in \mathbb{R}^n$, $|\hat{e}_i| = 1$

In simple terms, a partial derivative with respect to a specific variable x_i represents *how a functions rate of change along the direction of x_i changes over time.*

The Partial Derivative, Visualized



A few extra helpful videos (links): Khanacademy Video, Really Nice Animation

Computing Partial Derivatives

To compute partial derivatives, we treat all variables other than the variable we are differentiating with as **constants**, while letting all other derivative rules apply.

Example

$$\text{We have,} \quad f(x, y, z) = e^{y+z} sin(xy) + x^2 \cos(y)$$

$$\frac{\partial f}{\partial x} = e^{y+z} y \cos(xy) + 2x \cos(y)$$
$$\frac{\partial f}{\partial y} = e^{y+z} x \cos(xy) - x^2 \sin(y)$$
$$\frac{\partial f}{\partial z} = e^{y+z} \sin(xy)$$

Partial Derivatives Practice Problems (Part 1)

Compute the first and second partial derivatives of the following functions:

- $f(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}$
- $f(x, y) = \dfrac{\cos x^2}{y}$
- $f(x, y) = x^y$
- $f(x, y) = g_1(x)g_2(y)$ Use g_1' and g_2'.
- $f(x, y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$
- $f(x, y) = \ln(\sqrt{x^{\sin y}})$

Partial Derivatives Practice Problems (Part 2)

Problem 1) Show that $f(x,y) = \ln(\sqrt{(x-a)^2 + (y-a)^2})$ satisfies Laplace's Equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \vec{\nabla}^2 f = 0$$

Problem 2) Prove (using $\varepsilon - \delta$) that the following function is continuous around the point $(0,0)$ and demonstrate that the function has bounded partial derivatives $f_x(x,y)$ and $f_y(x,y)$, however this function is not differentiable at the point $(0,0)$.

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Problem 3) Show that the wave function $\Psi(x,t) = Ae^{i(kx-\omega t)}$ is a solution to the Schrödinger Equation, where $E_{tot} = \hbar\omega$, $E_{kin} = \frac{\hbar^2 k^2}{2m}$ and $E_{tot} = E_{kin} + E_{pot}$:

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \left[\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + E_{pot} \right] \Psi(x,t)$$

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Directional Derivatives

Until now, partial derivatives have told us the slope of the function along a specific direction. That is, along the coordinate axes. However, what if we want to describe the slope when travelling in a specific direction?

Imagine a hiker walking in a specific direction $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ through a mountain range characterized by the function $M(x, y)$. Then, the directional derivative describes the steepness value that the hiker climbs in that direction at each point. It is defined by taking each component of the direction vector, and scaling the partial derivative corresponding with its axis.

Definition: Directional Derivative

$$D_{\vec{v}} f(\vec{x}) = v_1 \cdot \frac{\partial f}{\partial x_1} + \cdots + v_i \cdot \frac{\partial f}{\partial x_i} + \cdots + v_n \cdot \frac{\partial f}{\partial x_n} = \vec{\nabla} f(\vec{x}) \cdot \vec{v}$$

The Nabla Operator $\vec{\nabla}$

The Nabla Operator $\vec{\nabla}$ is a very important operator in differential calculus, as it helps to extract very important information about multivariable functions. In a way, the nabla $\vec{\nabla}$ operator is a new type of derivative operator for multivariable functions, and can be applied in many ways.

Concept: The Nabla Operator $\vec{\nabla}$

The Nabla operator defines specific operations on multivariable functions and vector fields. It can be treated as a vector, in the sense that it can be defined as:

$$\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

Applications of $\vec{\nabla}$ (Gradient)

The **gradient** is the first way that the nabla $\vec{\nabla}$ operator is applied.

Definition: Gradient of a Multivariable Function

Given a multivariable **scalar function**: $f : \mathbb{R}^n \to \mathbb{R}$, the gradient of f is defined as:

$$\vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \quad \text{such that} \quad \vec{\nabla} f : \mathbb{R}^n \to \mathbb{R}^n$$

This means that the gradient operator maps a **scalar function** to a **vector-valued function** or **vector field**. The vectors of the gradient field represent the direction of steepest descent at a point, the magnitude of the vector representing the slope in the steepest direction.

The Gradient $\vec{\nabla} f$ (Visualized)

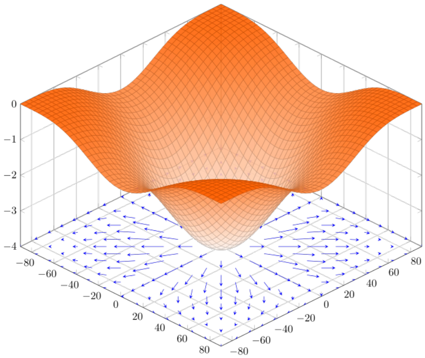

Figure: (Wikipedia) Surface and Gradient of $f(x, y) = -(\cos^2 x + \cos^2 y)^2$

A few extra helpful videos (links): Wikipedia, Youtube Video

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Total Differentials & Multivariable Chain Rule

Definition: Total Differential

Given a multivariable function $u = f(\vec{x})$, the total differential of u, du is defined as:

$$du = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

Defintion: Multivariable Chain Rule

Given a function $v = g(f_1(x, y), f_2(x, y))$, the total differential dv is defined as:

$$dv = \frac{\partial g}{\partial f_1} df_1 + \frac{\partial g}{\partial f_2} df_2 = \left(\frac{\partial g}{\partial f_1}\frac{\partial f_1}{\partial x} + \frac{\partial g}{\partial f_2}\frac{\partial f_2}{\partial x} \right) dx + \left(\frac{\partial g}{\partial f_1}\frac{\partial f_1}{\partial y} + \frac{\partial g}{\partial f_2}\frac{\partial f_2}{\partial y} \right) dy$$

Total Differentials & Multivariable Chain Rule: Practice

Compute the total differentials of the multivariable functions below:

- $u = f(\sin(xy), e^{x+y})$
- $f(x, y) = 2x^y$
- $v = f(g_1(x^2), g_2(\sqrt{y}))$
- $u = f(x, y, z), x = e^t, y = sin(t), z = ln(t)$
- $v = \psi(u, v), u = u(x, y), v = v(x, y)$
- $f_1(f_2(f_3(x, y, z)))$
- $a = g(x, y, z(x, y))$

Differential Equations

Concept: Ordinary Differential Equations

Differential equation are equations that involve specific quantities or functions $y(x)$ and their respective rates of change $y'(x), y''(x), \ldots$. An n-th order *explicit* differential equation can be expressed in the form:

$$F(x, y, y', \ldots, y^{(n-1)}) = y^{(n)}$$

Similarly, an n-th order *implicit* differential equation can be expressed as:

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

Not all ODEs are solvable analytically, e.g. y(x) is not always a closed expression. A common example of this is the second order ODE governing a simple-pendulum system.

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0$$

Exact Differentials (What does it mean to be exact?)

As you saw during the lecture, we can also write ODEs in the form:

$$A(x, y)dx + B(x, y)dy = 0$$

It was stressed that to solve for $y(x)$ the differential must be made exact, that is:

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

If we treat the left hand side as a total differential, $dF = A(x, y)dx + B(x, y)dy$ an inexact differential means that when we try to integrate this expression as a path integral, the result is **path dependent** (you will learn about this later in the course). In order to be able to integrate the function into a scalar field (which allows us to explicitly write some $F(x, y) = 0$) the differential must be exact, e.g. $\dfrac{\partial^2 F}{\partial x \partial y} = \dfrac{\partial^2 F}{\partial y \partial x}$. This is the rationale behind why we do this, and why an integrating factor makes it solvable.

Integrating Factor

Definition: Integrating Factor

Given an ODE of the form:

$$A(x, y)dx + B(x, y)dy = 0$$

$$\text{where } \frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x},$$

the **integrating factor** $\mu(x, y)$ is a factor such that when multiplied like below:

$$\mu(x, y)A(x, y)dx + \mu(x, y)B(x, y)dy = 0$$

$$\text{the equation becomes exact, e.g: } \frac{\partial}{\partial y}[\mu(x, y)A(x, y)] = \frac{\partial}{\partial x}[\mu(x, y)B(x, y)]$$

Integrating Factor: Practice Problems

Solve the following differential equations by finding a suitable integrating factor:

- $dy - (4y + e^{5x})dx = 0$
- $\dfrac{dy}{dx} + 3x^2y = x^2$
- $y' + \dfrac{1}{x}y = \dfrac{1}{x + x^3}$
- $y' + \dfrac{y}{x\ln(x)} = x, x > 1$
- $\dfrac{dy}{dx} + p(x)y = g(x)$

Integrating Factor: Practice Problems (cont'd)

Use the given forms of integrating factors to solve the expressions below (They may not all work!):

- $(3x+2y+y^2)dx+(x+4xy+5y^2)dy = 0, \quad \mu \equiv \mu(x+y^2), \mu \equiv \mu(x+y), \mu \equiv \mu(x^2+y)$

Multivariable Taylor Series

Concept: Multivariable Taylor Series - Nabla-Vector Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an infinitely differentiable multivariable function. Then one can approximate the function at a specific point in a neighbourhood about a point $\vec{a} \in \mathbb{R}^n$:

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h} + \frac{1}{2!}(\vec{h} \cdot \nabla)^2 f(\vec{a}) + \cdots + \frac{1}{n!}(\vec{h} \cdot \nabla)^n f(\vec{a}) + \mathcal{O}(|\vec{h}|^{n+1})$$

Alternatively:

$$f(\vec{a} + \vec{h}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\vec{h} \cdot \nabla)^n f(\vec{a})$$

This leads to some binomial expansion on the $(\vec{h} \cdot \nabla)^n$ term for 2-variable functions.

Tangent Planes

Using the gradient, one can construct a linear approximation to the function in the form of a co-dimension 1 hyperplane in $\mathbb{R}^n$ hyperspace. That is, a plane that is tangent to the surface $z = f(\vec{x})$.

Concept: Tangent Planes

- For 1-dim functions, the linear approximation of $f(x)$ at a point a is given by:

$$\ell(x) = \frac{df}{dx}(a) \cdot (x - a) + f(a)$$

- for n-dim functions, the linear approximation of $f(\vec{x})$ at a point $\vec{a}$ is given similarly by:

$$\ell(\vec{x}) = \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a}) + f(\vec{a})$$

Notice the parallels between the total derivative operator and the gradient!

Multivariable Taylor Expansion: Practice Problems

Problem 1: Expand the Taylor Series of $f(\vec{x})$ about a point $\vec{a}$ into an expression involving only partial derivatives and vector components. State the observed pattern and produce a general formula.

Problem 2: Compute the following multivariable Taylor Series up to the second order around a small neighbourhood at 0.

- $e^{\sin(xy)}$
- $e^x \cos y$
- e^{xyz} (Only compute first order)

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Tangent Planes: Practice Problems

Problem 1: Compute the tangent plane at point $\vec{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$ for the following multivariable functions:

- $z = x^2 + y^2$
- $z = 3e^{-2(x^2+y^2)}$
- $z = \frac{sin(x^2+y^2)}{x^2+y^2}$

Problem 2: Find the normal vector to the following surfaces at point $\vec{0}$

- $f(x, y) = (x - a)^2 - (y - b)^3$
- $f(x, y, z) = \frac{sin(x)cos(y)}{sin(z)+1}$

Problem 3: Prove that the normal vector of a hemisphere is given by: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \frac{1}{z}$, where

$$z = \sqrt{1 - x^2 + y^2}$$

Maxima & Minima of Multivariable Functions: Definition 1

Definition: Local Maximum/Minimum of a multivariable function

Given a multivariable function $f : \mathbb{R}^n \to \mathbb{R}$, a point $\vec{a}$ is called a local maximum if there exists a neighbourhood of $\vec{a}$, $U(\vec{a}) \subset \mathbb{R}^n$ such that:

$$\forall \vec{x} \in U(\vec{a}), \quad f(\vec{x}) \leq f(\vec{a})$$

Equivalently, a point $\vec{a}$ is a minimum if $\exists U(\vec{a}) \subset \mathbb{R}^n$ such that:

$$\forall \vec{x} \in U(\vec{a}), \quad f(\vec{x}) \geq f(\vec{a})$$

Maxima & Minima of Multivariable Functions: Definition 2

Concept: Maxima/Minima from Gradient ∇

At a maximum or a minimum, the tangent plane to the surface formed by $z = f(x, y)$ would be exactly horizontal. E.g, of the form $z = c, c \in \mathbb{R}$. In order for the plane to be only a constant, the tangent plane then must take this form:

$$\ell(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) \overset{!}{=} c, c \in \mathbb{R}$$

$$\implies \nabla f(\vec{a})(\vec{x} - \vec{a}) = 0, \forall \vec{x} \implies \boxed{\nabla f(\vec{a}) = 0}$$

This is just one way to obtain this result.

After finding our stationary points, one must further characterize their concavity to determine whether the point is a **minima, maxima or saddle point.**

Characterizing Maxima and Minima: Part 1

Example

We find maxima and minima by looking for when the gradient is zero: $\nabla f(\vec{a}) = 0$. To determine the concavity of function near the extrema, we can use **Lagrange's Method**:

$$\text{We define:} \quad D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$$

$$\begin{cases} \textbf{Minimum} & \text{if } \partial_{xx} f(\vec{a}) > 0, \quad D(\vec{a}) > 0 \\ \textbf{Maximum} & \text{if } \partial_{xx} f(\vec{a}) < 0, \quad D(\vec{a}) > 0 \\ \textbf{Saddle Point} & \text{if } \quad D(\vec{a}) < 0 \end{cases}$$

This follows from examining conditions on the second order term of the multivariable taylor series and its positive or negative definiteness.

Characterizing Maxima and Minima: Part 2

Example

An alternative way to analyze the convexity of a multivariable function is to consult the eigenvalues of the **Hessian Matrix**:

$$\text{We define:} \quad \mathcal{H} = \begin{pmatrix} \partial_{xx} f & \partial_{xy} f \\ \partial_{yx} f & \partial_{yy} f \end{pmatrix}$$

$$\begin{cases} \textbf{Minimum} & \text{if } \lambda_i > 0, i = 1, 2 \\ \textbf{Maximum} & \text{if } \lambda_i < 0, i = 1, 2 \\ \textbf{Saddle Point} & \text{if } \lambda_1 < 0, \lambda_2 > 0 \end{cases}$$

This follows from the fact that the eigenvalues tell us information about the definiteness of the Hessian matrix.

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Maxima & Minima: Practice Problems

Problem 1: Calculate the critical points of each of the following functions, and determine whether it is a maximum, minimum, or saddle point:

- $f(x, y) = \frac{x}{2} + e^{-(x^2+y^2)}$
- $f(x, y) = (y-2)x^2 - y^2$
- $z = 7x - 8y + 2xy - x^2 + y^3$

Problem 2: Calculate the equation that describes the global minima of this function.

$$x^2 + y^2 + 3e^{-(x^2+y^2)}$$

Constrained Optimization (Lagrange Multiplier Method)

We will now consider the class of optimization problem where constraints are placed on x and y. The problem is as follows.

Concept: Constrained System

Let $f(\vec{x})$ be a multivariable function, $\vec{x} \in \mathbb{R}^n$ which we seek to optimize with repect to the constraint $g(\vec{x}) = 0$. Such constraints are called homoclinic constraints. Optimizing this function, we introduce the Lagrange Function, which is defined as:

$$\mathcal{L}(\vec{x}, \lambda) = f(\vec{x}) - \lambda(g(\vec{x}))$$

We now optimize this by setting the gradient of the Lagrange function to zero.

$$\nabla \mathcal{L}(\vec{x}, \lambda) = \nabla f(\vec{x}) - \nabla \lambda(g(\vec{x})) = 0$$

$$\implies \nabla f(\vec{x}) = \nabla \lambda(g(\vec{x}))$$

Solving this system of $n + 1$ equations, we find our extremum values.

Lagrange Multipliers: Practice Problems

1. Use Lagrange Multipliers to find the global maxima and minima of $f(x, y) = x^2 + 2y^2 - 4y$ subject to the constraint $\sqrt{x^2 + y^2} = 3$. After this, compute the maximum value of the function subject to the inequalty constraint $x^2 + y^2 \leq 4$.
2. Find all local extrema of the function $f(x, y) = e^{\frac{x+y}{2}}(x^2 - 2y^2)$
3. Find the maxima and minima of $f(x, y, z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2 = 1$.
4. Let $h(x, y) = 5 - 3x - 4y$. Optimize with respect to the constraint $x^2 + y^2 = 25$

Vector Fields

Definition: Vector Field & Scalar Field

A vector field, also called a vector valued function $\vec{f}(\vec{x})$, is defined as a vector of scalar function components, e.g:

$$\vec{f}(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{pmatrix}$$

Formally, it is in most cases a mapping $\vec{f} : \mathbb{R}^n \to \mathbb{R}^n$

A scalar field $g(\vec{x})$ is what many are familiar with. It is a mapping from a vector space to a scalar field, $g : \mathbb{R}^n \to \mathbb{R}$

Differential Operations on Vector Fields: Divergence & Curl

The Nabla Operator ∇ can be applied to more objects than just a scalar field, for example **vector fields**. Recall that there are two vector products that we mainly concern ourselves with: the **dot product** and the **cross product**. Applying the nabla operator to a **vector field** with a dot product or cross product tells us specific information about the vector fields.

$$\underbrace{\nabla \cdot \vec{f}}_{\text{Divergence}} \qquad \underbrace{\nabla \times \vec{f}}_{\text{Curl}}$$

The **divergence** tells us how much the vector field emerges or converges to points. Formally, it is the flux of an infinitesimal hypervolume element.
The **curl** of a vector field tells us how much the vector circulates around a specific point.

Application of Divergence

Example

Given a vector-valued function $\vec{f}(\vec{x})$, its divergence is defined as:

$$\nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}$$

Applications of Curl

Example

Given a vector-valued function $\vec{f}(\vec{x})$, its curl is defined as:

$$\nabla \times \vec{f} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

Vector Differential Operators: Practice Problems

- Prove that the divergence of the curl of an arbitrary vector field is zero.
- Prove that the curl of the gradient of an arbitrary scalar field is zero.
- Compute the curl of the vector field:

$$\vec{F} = \left(4y^2 + \frac{3x^2 y}{z^2}\right)\vec{i} + \left(8xy + \frac{x^3}{z^2}\right)\vec{j} + \left(11 - \frac{2x^3 y}{z^3}\right)\vec{k}$$

  These types of vector fields are called conservative vector fields as they are defined by a potential function. E.g: $F = \nabla A$. Gravitational potential is an example of a conservative vector field.

- Compute the divergence of $\vec{F}(x, y) = \langle x^2 y, \, y - xy^2 \rangle$ and of $\vec{F}(x, y) = \langle x^2 y, \, 5 - xy^2 \rangle$. Which vector field is incompressible?

Vector Differential Operators: Practice Problems

- Compute the Laplacian $(\nabla \cdot \nabla)$ of the following scalar function $f(x, y, z) = e^{\sin(xy)} + \cos(zy)$

- Prove that the graviational force field is conservative:

$$\vec{F}(x, y, z) = -Gm_1 m_2 \left\langle \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle.$$

Line Integrals: A Quick Overview

In the past we have integrated multivariable functions over an **area** domain, but what if we wish to integrate a multivariable function $f(\vec{x})$ along a particular curve $\vec{r}(t)$?

For this problem, we consult **line integrals**, which evaluate the integral of a function along a specific path in the input space.

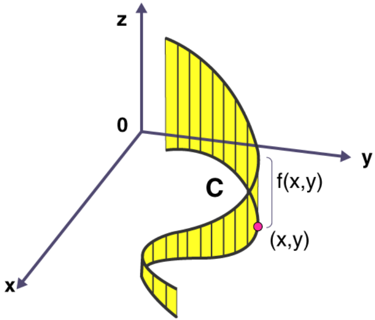

Figure: Line Integral over a function $f(x, y)$

Line Integrals: Definitions

Definition: Line Integral

Given a multivariable function $f(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$. The line integral of $f(\vec{x})$ over a curve C is denoted as:

$$\int_C f(x, y) d\ell$$

where $d\ell$ represents the infinitesimal of the curve C.

Line Integrals Conceptualized: Part 1

Concept: Line Integral

As we learned earlier, one can parameterize a curve in $\mathbb{R}^n$ with a **single parameter** t, e.g:

$$\vec{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

As of such, **infinitesimals** of this curve $d\ell$ are given by infinitesimal steps in each coordinate direction, e.g:

$$d\ell = \sqrt{dx_1^2 + dx_2^2 + \cdots + dx_n^2} = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2} dt = ||\vec{r}'(t)|| dt$$

Line Integrals Conceptualized: Part 2

Change of Variables: Partial Differential Equations

Concept: Change of Variables in Partial Differential Equations

A change of variables can be defined as a transformation in input space. For typical functions, cartesian coordinates $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$ are used.

When performing a change of variables, this input vector is changed to be expressed as:

$$\vec{x} = \langle x_1(u_1, \ldots, u_n), x_2(u_1, \ldots, u_n), \ldots, x_n(u_1, \ldots, u_n) \rangle$$

$$\downarrow$$

$$\vec{u} = \langle u_1(x_1, \ldots, x_n), u_2(x_1, \ldots, x_n), \ldots, u_n(x_1, \ldots, x_n) \rangle$$

By changing the independent variables from $x_1, \ldots x_n$ to $u_1, \ldots, u_n$, the differentials of functions of these variables also change, and in turn, their partial and total derivatives.

Change of Variables: Partial Differential Equations

Change of Variables in Partial Differential Equations

When a change of variables occurs for a function $z : \mathbb{R}^n \to \mathbb{R}$, $z \cong z(\vec{x})$, its input is reexpressed in terms of the new independent variables $\vec{u}$, where $\vec{x} \cong \vec{x}(\vec{u})$.

To fully transform a partial differential equation, its partial derivatives with respect to the old variables are reexpressed in the new variables $\vec{u}$. Below is an example for $z \cong z(x, y)$:

Let $x \cong x(u, v), y \cong y(u, v)$, then $\boxed{z(x(u, v), y(u, v)) \rightsquigarrow z(u(x, y), v(x, y))}$:

$$\frac{\partial z}{\partial x} = \frac{\partial z(x(u, v), y(u, v))}{\partial x} = \frac{\partial z(u(x, y), v(x, y))}{\partial x} \underbrace{=}_{\text{Chain Rule}} \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

Note that this means that all changes of variables have to be **invertible**, e.g. bijective. We can check this by ensuring that the determinant of the Jacobian is non-singular ($\neq 0$).

PDE Change of Variables: Investigative Problem (Useful)

Given a function $F(x, y)$, calculate its first order and second order partial derivatives with respect to x and y subject to new independent variables u and x where $u = u(x, y)$ and $v = v(x, y)$. E.g. find:

$$\frac{\partial F}{\partial x} = ?, \frac{\partial F}{\partial y} = ?, \frac{\partial^2 F}{\partial x^2} = ?, \frac{\partial^2 F}{\partial y^2} = ?$$

Solving this problem will allow you to devise a general formula to use and substitue into derivatives of F when solving PDEs by method of change of variables.

PDE Change of Variables: Practice Problems

For the following problems, perform the stated change of variables:

- Let $u = xy, v = \frac{x}{y}$, and solve $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x^2$
- Let $v = u - c, c \in \mathbb{R}$, and find w/o solving $\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = 0$
- Transform Laplace's Equation $\Delta u = 0$ to polar coordinates.
- Solve the PDE $x\partial_x u - y\partial_y u + u = 2$ using the substitution $a = xy,\ b = x$.

Change of Variables: Ordinary Differential Equations

Concept: Change of Variables in ODEs

Changing variables in ODEs is quite straightforward. Assuming that we would like to change a function $y(x) \rightsquigarrow y(t)$ through a change of independent variables $x \cong x(t)$, we can express the derivatives of y under the changed variable as follows:

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$$

$$y'' = \frac{d}{dx}\frac{\dot{y}}{\dot{x}} = \frac{1}{\dot{x}}\frac{d}{dt}\frac{\dot{y}}{\dot{x}} = \frac{1}{\dot{x}}\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{y}^2} =$$

ODE Change of Variables: Investigative Problem

Given a function $y(x)$, calculate its first order and second order total derivatives with respect to x (y' and y'') with respect to a new variable t defined by the change of variables $x = x(t)$.

$$\frac{dy}{dx} = ? \quad \frac{d^2y}{dx^2} = ?$$

What if we took the function y as a new independent variable such that $x = x(y)$ how would the derivatives of y w.r.t x be expressed when y becomes the variable and x becomes the function?

Infinite Series Introduction

Given a infinite sequence $(a_n)_{n=1}^{\infty}$, its corresponding infinite series is defined as $\sum_{n=1}^{\infty} a_n$. Infinite series only take on finite values if they are **convergent**. Otherwise, they diverge to $\pm\infty$.

There are many tests that have been develop over history to determine if a sequence is convergent. This section concerns evaluating the values for infinite series and determining convergence.

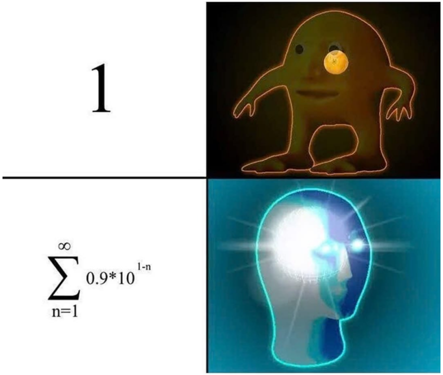

Figure: $0.99\overline{99} = 1$.

Infinite Series Convergence Tests: Comparison Test

Definition: Comparison Test

Given two infinite series $\sum a_n$ and $\sum b_n$:

$$\text{if } 0 < a_n < b_n, \text{ and } \sum b_n \text{ converges, then so does } \sum a_n$$

Similarly, given two infinite series $\sum a_n$ and $\sum b_n$:

$$\text{if } 0 < b_n < a_n \text{ and } \sum b_n \text{ diverges, then so does } \sum a_n$$

Infinite Series Convergence Tests: Ratio Test

Definition: Ratio Test (D'Alembert)

This test considers the limit $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$.

$$\text{If } L < 1 \implies \text{the series } \sum a_n \text{ converges.}$$

$$\text{If } L > 1 \implies \text{the series } \sum a_n \text{ diverges.}$$

$$\text{If } L = 1 \implies \text{the test is inconclusive.}$$

Infinite Series Convergence Tests: Root Test

Definition: Ratio Test (D'Alembert)

This test considers the limit $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

$$\text{If } L < 1 \implies \text{the series } \sum a_n \text{ converges.}$$

$$\text{If } L > 1 \implies \text{the series } \sum a_n \text{ diverges.}$$

$$\text{If } L = 1 \implies \text{the test is inconclusive.}$$

Infinite Series Convergence Tests: Integral Test

Definition: Integral Test

Given an infinite sum $\sum a_n$, we consider the integral of $f(n) = a_n$:

$$\int_1^\infty f(n)dn = \lim_{k\to\infty} \int_1^k f(n)dn$$

If the integral $\displaystyle\int_1^\infty f(n)dn < \infty$ then the series $\sum a_n$ converges, and if $\displaystyle\int_1^\infty f(n)dn$ diverges, $\sum a_n$ diverges similarly.

Infinite Series Convergence Tests: Reference Series

Definition: Reference Series Test

Some parameterized infinite sums have well documented convergence criterion. An example of the reference series test is the inverse x power sequence.

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \implies \begin{cases} \alpha > 1 \implies \text{converges} \\ \alpha \leq 1 \implies \text{diverges} \end{cases}$$

Infinite Series Convergence tests: Cauchy's Convergence Test

Definition: Cauchy's Convergence Test

An infinite series $\sum a_n$ is convergent if and only if $\forall \varepsilon > 0$, there is a natural number N such that:

$$|s_{n+p} - s_n| = |a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \varepsilon$$

holds for all $n > N$ and all $p \geq 1$.

In other words, the series converges if its **partial sums s_n form a Cauchy Sequence**.

$$s_n = \sum_{i=0}^{n} a_i \quad \text{(partial sum)}$$

Infinite Series Convergence Tests: Alternating Series (Leibniz' Rule)

Definition: Leibniz' Rule for Alternating Series

For a sum $\sum a_n$ where $\lim_{n\to\infty} a_n = 0$, a_n is monotonic, and $a_n > 0$, then:

$$\sum_{n=1}^{\infty} (-1)^n a_n \text{ converges always.}$$

This theorem can be proved using Cauchy's Convergence Test and analyzing the partial sums. Consider for example the difference of partial sums s_6, s_{12} and s_6, s_{11}. Use the inequalities that arise to create a generalization.

Infinite Series Convergence Tests: Dirichlet

Definition: Dirichlet Test for Series Convergence

For a series $\sum a_n$, if a_n is monotonic, and $\lim_{n\to\infty} a_n = 0$, if:

$$|\sum_{n=1}^{N} b_n| \leq M, \forall N \in \mathbb{N}, M \in \mathbb{R}^+ \implies \sum_{n=1}^{\infty} a_n b_n \text{ converges.}$$

Infinite Series Convergence Tests: Abel's Test

Definition: Abel's Test for Series Convergence

If an infinite series $\sum a_n$ converges, and b_n is a monotonic, bounded sequence, then:

$$\sum_{n=1}^{\infty} a_n b_n \text{ converges.}$$

Infinite Series Convergence Tests: Practice Problems

For each of the following series, determine if the series converges or diverges using one of the aforementioned tests.

- $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12}$
- $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(\frac{\pi n}{2})}{n}$
- $\displaystyle\sum_{n=0}^{\infty} \frac{n^2-1}{n^2-n}$
- $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2(n)}$
- $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{5^{n+1}}$

The End